A. Proofs

We first present the following lemma in order to prove Proposition 1:

**Lemma 1.** Define

\[
\pi^\gamma(p, r) = p((a_1 + a_2) - (b_1 + b_2)p + \gamma(r - p)) \quad \text{and} \\
\pi^\lambda(p, r) = p((a_1 + a_2) - (b_1 + b_2)p + \lambda(r - p)).
\]

Assume the firm sells to both segments. Then

\[
p_a(r) \triangleq \arg\max_p \{ p((a_1 + a_2) - (b_1 + b_2)p + \gamma(r - p)^+ - \lambda(p - r)^+) \} = \begin{cases} 
  p^\gamma(r) & \text{if } r \geq r^\gamma, \\
  r & \text{if } r^\lambda \leq r \leq r^\gamma, \\
  p^\lambda(r) & \text{if } r \leq r^\lambda; 
\end{cases}
\]

(1)

**Proof of Lemma 1.** Consider the objective function

\[
f(p, r) \triangleq -p((a_1 + a_2) - (b_1 + b_2)p + \min\{\gamma(r - p), \lambda(r - p)\}).
\]

The original problem is equivalent to finding the global minimizer of \(f(p, r)\) when given \(r \geq 0\). It is easy to verify that \(f(p, r)\) is convex in \(p\). Because \(f\) is not differentiable at \(p = r\), we consider its subdifferential \(\partial_p f(p, r)\):

\[
\partial_p f(p, r) = \{ c \mid f(p + \Delta p, r) - f(p, r) \geq c\Delta p \ \forall \Delta p \}.
\]

In other words, \(\partial_p f(p, r)\) is the set of the slopes of all lines that (a) pass through \((p, f(p, r))\) and (b) lie below the graph of \(f(p, r)\). For differentiable points, \(\partial_p f(p, r)\) is simply its derivative. Observe that

\[
\partial_p f(p, r) = \begin{cases} 
  (a_1 + a_2 + \gamma r) - 2(b_1 + b_2 + \gamma)p & \text{if } p > r, \\
  (a_1 + a_2 + \lambda r) - 2(b_1 + b_2 + \lambda)p & \text{if } p < r, \\
  \{(a_1 + a_2 + \xi r) - 2(b_1 + b_2 + \xi)p \mid \xi \in [\gamma, \lambda] \} & \text{if } p = r.
\end{cases}
\]

By the definition of a subdifferential, \(p_a(r)\) is a global minimizer of \(f(p, r)\) if and only if \(0 \in \partial_p f(p, r)\).
∂p_f(p_a(r), r). The equivalent formal expression is

\[
p_a(r) = \begin{cases} 
\frac{a_1 + a_2 + \gamma r}{2(b_1 + b_2 + \gamma)} & \text{if } \frac{a_1 + a_2 + \gamma r}{2(b_1 + b_2 + \gamma)} > r, \\
\frac{a_1 + a_2 + \gamma r}{2(b_1 + b_2 + \gamma)} & \text{if } \frac{a_1 + a_2 + \gamma r}{2(b_1 + b_2 + \gamma)} < r, \\
r & \text{if } (a_1 + a_2 + \xi r) - 2(b_1 + b_2 + \xi) r = 0 \text{ for some } \xi \in [\gamma, \lambda]; 
\end{cases}
\]

or

\[
p_a(r) = \begin{cases} 
p^\gamma(r) & \text{if } r > r^\gamma, \\
p^\lambda(r) & \text{if } r < r^\lambda, \\
r & \text{if } r \in [r^\lambda, r^\gamma]. 
\end{cases}
\]

This completes the proof.

**Proof of Proposition** To prove this proposition, we first establish the following lemma.

**Lemma 2.** Assume that Assumption holds and that segment-2 customers are loss neutral (i.e., \( \gamma = \lambda \)). Then

\[
\arg\max_{p \in [0, a_1/b_1]} \{ \Pi(p, r) \} = \begin{cases} 
\frac{a_1}{2b_1} & \text{if } 0 \leq r < \min \{ \bar{r}, \frac{a_1}{b_1} \}, \\
\frac{a_1 + a_2 + \gamma r}{2(b_1 + b_2 + \gamma)} & \text{if } \max \{ \bar{r}, 0 \} \leq r \leq \frac{a_1}{b_1}; 
\end{cases}
\]

in this expression, \( \bar{r} = \frac{a_1}{\gamma} \sqrt{\frac{b_1 + b_2 + \gamma}{b_1} - \frac{a_1 + a_2}{\gamma}} \).

**Proof.** For given \( p \) and for \( r \in [0, a_1/b_1] \), we have

\[
\Pi(p, r) = pD(p, r) = \begin{cases} 
-(b_1 + b_2 + \gamma)p^2 + (a_1 + a_2 + \gamma r)p & \text{if } 0 \leq p \leq \frac{a_2 + \gamma r}{b_2 + \gamma}, \\
-b_1 p^2 + a_1 p & \text{if } \frac{a_2 + \gamma r}{b_2 + \gamma} < p \leq \frac{a_1}{b_1}. 
\end{cases}
\]

It is obvious that \( \arg\max_p \{ p(a_1 - b_1 p) \} = a_1 / 2b_1 \) and that \( \max_p \{ p(a_1 - b_1 p) \} = a_1^2 / 4b_1 \). Furthermore,

\[
\arg\max_p \left\{ p(a_1 + a_2 - (b_1 + b_2)p + \gamma(r - p)) \right\} = \frac{a_1 + a_2 + \gamma r}{2(b_1 + b_2 + \gamma)}
\]

and so

\[
\max_p \left\{ p(a_1 + a_2 - (b_1 + b_2)p + \gamma(r - p)) \right\} = \frac{(a_1 + a_2 + \gamma r)^2}{4(b_1 + b_2 + \gamma)}.
\]

The two local maxima are equal for all values of \( r \) that solve \( \frac{a_1}{4b_1} = \frac{(a_1 + a_2 + \gamma r)^2}{4(b_1 + b_2 + \gamma)} \), which admits two solutions. The first solution, \( r_s \), satisfies \( \frac{a_1}{2\sqrt{b_1}} = -\frac{a_1 + a_2 + \gamma r_s}{2\sqrt{b_1} + b_2 + \gamma} \); the second solution, \( \bar{r} \), satisfies

\[
\frac{a_1}{2\sqrt{b_1}} = \frac{a_1 + a_2 + \gamma r}{2\sqrt{b_1} + b_2 + \gamma}.
\]

Observe that \( r_s = -\frac{a_1}{\gamma} \sqrt{\frac{b_1 + b_2 + \gamma}{b_1} - \frac{a_1 + a_2}{\gamma}} \) is negative and less than \( \bar{r} \). By the properties of quadratic functions, if \( r > \max \{ \bar{r}, r_s \} \) then \( \frac{a_1}{4b_1} < \frac{(a_1 + a_2 + \gamma r)^2}{4(b_1 + b_2 + \gamma)} \), and if \( r \) is between \( \bar{r} \) and \( r_s \) then
\[ \frac{a_1^2}{4b_1} > \frac{(a_1+a_2+\gamma r)^2}{4(b_1+b_2+\gamma)}. \] Therefore,

\[
\arg\max_{p} \{\Pi(p, r)\} = \begin{cases} \frac{a_1}{2b_1} & \text{if } \frac{a_1^2}{4b_1} > \frac{(a_1+a_2+\gamma r)^2}{4(b_1+b_2+\gamma)}, \\ \frac{a_1+a_2+\gamma r}{2(b_1+b_2+\gamma)} & \text{if } \frac{a_1^2}{4b_1} \leq \frac{(a_1+a_2+\gamma r)^2}{4(b_1+b_2+\gamma)}, \\ \frac{a_1}{2b_1} & \text{if } 0 < r < \min \{\bar{r}, \frac{a_1}{b_1}\}, \\ \frac{a_1+a_2+\gamma r}{2(b_1+b_2+\gamma)} & \text{if } \max\{\bar{r}, 0\} \leq r \leq \frac{a_1}{b_1}. \end{cases}
\]

This completes the proof of Lemma 2.

We now proceed with the proof of Proposition 1. Our first task is to show that \(\bar{r}^\lambda, \bar{r}^\gamma,\) and \(\bar{r}^0\) each satisfy the following statements. For \(r \geq 0,\)

\[
p^\lambda(r)(a_1 + a_2 - (b_1 + b_2)p^\lambda(r) + \lambda(r - p^\lambda(r))) < \Pi_1\left(\frac{a_1}{2b_1}\right) \iff r < \bar{r}^\lambda, \tag{2}
\]

\[
p^\gamma(r)(a_1 + a_2 - (b_1 + b_2)p^\gamma(r) + \gamma(r - p^\gamma(r))) < \Pi_1\left(\frac{a_1}{2b_1}\right) \iff r < \bar{r}^\gamma; \tag{3}
\]

\[
\{r \in (\bar{r}^0, r^\lambda) : \Pi_1\left(\frac{a_1}{2b_1}\right) \iff r < \bar{r}^\gamma. \tag{4}
\]

Here we prove only \(\bar{r}^\lambda\) because \(\bar{r}^\gamma\) and \(\bar{r}^0\) follow directly from Lemma 2. Note that \(r(a_1 + a_2 - (b_1 + b_2)r) = \Pi_1\left(\frac{a_1}{2b_1}\right)\) has two solutions: \(r^0,\) which is less than \(\frac{a_1+a_2}{2(b_1+b_2)};\) and \(r,\) which is greater than that fraction. Equation (4) then follows because \(r^\gamma < \frac{a_1+a_2}{2(b_1+b_2)}\).

We can now show case 1 of the proposition. Since \(\bar{r}^0 \leq r^\lambda,\) it follows from (4) that \(\Pi(r^\lambda, r^\lambda) \geq \Pi_1(a_1/2b_1).\) And because \(r^\lambda = p^\lambda(r^\lambda),\) we have \(\Pi(p^\lambda(r^\lambda), r) \geq \Pi_1(a_1/2b_1).\) This inequality implies that, by (2), \(\bar{r}^\lambda \leq r^\lambda.\) At this point, optimizing \(\Pi(p, r)\) requires only that we compare the two local maxima: one maximum at price \(p_0(r),\) which maximizes revenue from the aggregated market; and one maximum at \(a_1/2b_1,\) which maximizes only segment-1 revenue.

If \(r \leq \bar{r}^\lambda \leq r^\lambda\) then, by (1), \(p_0(r) = \frac{a_1+a_2+\lambda r}{2(b_1+b_2+\lambda r)}\) in the aggregated market. By Lemma 2, \(\Pi_1\left(\frac{a_1}{2b_1}\right) \geq \Pi\left(\frac{a_1+a_2+\lambda r}{2(b_1+b_2+\lambda r)}, r\right).\) Therefore, \(p_m(r) = \frac{a_1}{2b_1}.\) Similarly, if \(r \in (r^\lambda, r^\gamma)\) then, by Lemma 2, \(\Pi_1\left(\frac{a_1}{2b_1}\right) < \Pi\left(\frac{a_1+a_2+\lambda r}{2(b_1+b_2+\lambda r)}, r\right).\) As a consequence, \(p_m(r) = \frac{a_1+a_2+\lambda r}{2(b_1+b_2+\lambda r)}\). For \(r \in [r^\lambda, a_1/b_1],\) we have

\[
\Pi(p_m(r), r) \geq \Pi\left(\frac{a_1+a_2+\lambda r}{2(b_1+b_2+\lambda r)}, r\right) > \Pi_1\left(\frac{a_1}{2b_1}\right). \tag{5}
\]

It follows that \(p_m(r) = p_0(r)\) for \(r \in [r^\lambda, a_1/b_1].\)

For case 2 in Proposition 1 by the definition of \(\bar{r}^0\) we have \(\Pi(p_0(\bar{r}^0), \bar{r}^0) = \Pi(\bar{r}^0, \bar{r}^0) = \Pi_1(a_1/2b_1).\) Then \(\Pi(p_0(r), r) \leq \Pi_1(a_1/2b_1)\) for \(r < r^0\) because \(\Pi(p, r)\) is an increasing function of \(r\) for any given \(p.\) We therefore have that \(p_m(r) = a_1/2b_1\) for \(r \in [0, r^0]\) and that \(p_m(r) = p_0(r)\) for \(r \in [r^0, a_1/b_1].\)

The proof for case 3 is similar to that for case 1 and so has been omitted.

Proof of Proposition 3 Because \(p_0(r) \leq p_0\left(\frac{a_1}{b_1}\right) = p^\gamma\left(\frac{a_1}{b_1}\right) = \frac{a_1+a_2+\gamma(a_1/b_1)}{2(b_1+b_2+\gamma)} < \frac{a_1}{2b_1},\) there is a downward jump at the discontinuity of \(p_m(r)\) in all three cases of Proposition 1.

In case 1 of that proposition, note that \(\bar{r}^\lambda \leq r^\lambda = \frac{a_1+a_2}{\lambda+2(b_1+b_2)} < \frac{a_1}{2b_1}.\) Hence the equation \(p_m(r) = \frac{a_1}{2b_1} = r\) has no solutions for \(r < \bar{r}^\lambda.\) For \(r \geq \bar{r}^\lambda,\) the equation \(p_m(r) = p_0(r) = r\) has
solutions \( r \in [r^\lambda, r^\gamma] \); this scenario corresponds to case (i) in Proposition 2. Similarly, for case 2 in Proposition 1, any \( r \in [\theta, r^\gamma] \) is a solution to \( p_m(r) = r \); this scenario corresponds to case (ii) in Proposition 2.

For case 3 in Proposition 1 there are two possibilities. If \( r^\gamma \geq a_1/2b_1 \), then \( p_m(r) = r \) has one solution: \( a_1/2b_1 \). Otherwise, \( p_m(r) = r \) has no solutions. Those two possibilities are illustrated by panels (c) and (d), respectively, in Figure 1 which correspond to the respective cases (iii) and (iv) of Proposition 2.

**Proof of Lemma 3.** For cases (i), (ii), and (iii) of Proposition 2 there exist steady states. In all three cases, \( p_m(\cdot) \) consists of piecewise linear functions and has at most one discontinuity, and there is no linear segment whose slope exceeds 1. We use case (i) in Proposition 2 to show that \( r_t \) and \( p_m(r_t) \) converge to one steady state; the claims for cases (ii) and (iii) follow similarly. When \( r_t < r^\lambda \), we must have \( r_{t+s} \geq r^\lambda \) for some \( s > 0 \) because \( r_{t+1} - r_t = (1 - \theta)(p_m(r_t) - r_t) \geq (1 - \theta)(p_m(r_t) - r^\lambda) > 0 \) is bounded away from zero.

When \( r^\lambda > r_t \geq r^\gamma \), we can show that \(|r_{t+1} - r^\lambda| = \theta|r_t - r^\lambda| + (1 - \theta)|p_m(r_t) - r^\lambda| = (\theta + (1 - \theta)\frac{a_1}{b_1 + b_2 + \lambda})|r_t - r^\lambda|\) is a contraction. Therefore, \( r_t \) converges to \( r^\lambda \) and so does \( p_m(r_t) \).

When \( r^\lambda \leq r_t \leq r^\gamma \), the reference point \( r_t \) is already a steady state. When \( r_t > r^\gamma \), we show that \( r_t \) converges to \( r^\gamma \) much as when \( r^\lambda > r_t \geq r^\gamma \). As a result, \( r_t \) and \( p_m(r_t) \) always converge to a steady state. The convergence for case (i) is illustrated in panel (a) of Figure 1. Cases (ii) and (iii) of Proposition 2 can be proved similarly and are presented graphically in, respectively, panels (b) and (d) of Figure 1.

Finally, for case (iv) in Proposition 2 there is no steady state. The trajectory of \( r_t \) converges to a stable cycle, and so does that of \( p_m(r_t) \) (see Gardini and Tramontana 2010, Sec. 5.1). This completes the proof.

Before proving Proposition 4, we show the following lemma.

**Lemma 3.** For any \( r \in [0, a_1/b_1) \) and \( 0 < \Delta r < a_1/b_1 - r \), we have \( 0 \leq \frac{V(r + \Delta r) - V(r)}{\Delta r} \leq \frac{\lambda a_1}{(1 - \beta \theta)b_1} \).

**Proof of Lemma 3.** Define the value iteration

\[
V^{(0)}(r) \equiv 0, \\
V^{(k+1)}(r) = \max\{\Pi(p, r) + \beta V^{(k)}(\theta r + (1 - \theta)p)\}, \quad k = 1, 2, \ldots 
\]

Clearly, \( V(r) = \lim_{k \to \infty} V^{(k)}(r) \). We will prove the lemma by induction. We have \( V^{(0)}(r + \Delta r) - V^{(0)}(r) = 0 \) for \( k = 0 \); for \( k \geq 0 \),

\[
V^{(k+1)}(r + \Delta r) - V^{(k+1)}(r) = \max_{p} \left\{ \Pi(p, r + \Delta r) + \beta V^{(k)}(\theta (r + \Delta r) + (1 - \theta)p) \right\} \\
- \max_{p} \left\{ \Pi(p, r) + \beta V^{(k)}(\theta r + (1 - \theta)p) \right\}.
\]

Because \( \Pi(p, r) \) and \( V^{(k)}(\theta r + (1 - \theta)p) \) are both increasing in \( r \), it must be that \( V^{(k+1)}(r + \Delta r) - V^{(k+1)}(r) \geq 0 \). We need to show that \( V^{(k+1)}(r + \Delta r) - V^{(k+1)}(r) \leq \frac{\lambda a_1}{(1 - \beta \theta)b_1} \Delta r \). Toward that end, we write

\[
\max_{p} \left\{ \Pi(p, r + \Delta r) + \beta V^{(k)}(\theta (r + \Delta r) + (1 - \theta)p) \right\} - \max_{p} \left\{ \Pi(p, r) + \beta V^{(k)}(\theta r + (1 - \theta)p) \right\} \\
\leq \max_{p} \left\{ \Pi(p, r + \Delta r) - \Pi(p, r) + \beta (V^{(k)}(r + \Delta r) + (1 - \theta)p) - V^{(k)}(\theta r + (1 - \theta)p) \right\}.
\]
The function $\Pi(p,r)$ has one of the following three forms: $p((a_1 + a_2) - (b_1 + b_2)p + \gamma(r - p))$, $p((a_1+a_2)-(b_1+b_2)p+\lambda(r-p))$, or $p(a_1-b_1p)$. Therefore, $\Pi(p,r+\Delta r)-\Pi(p,r) \leq \lambda p \Delta r \leq \lambda a_1 \Delta r/b_1$.

It now follows from the inductive hypothesis that

$$V^{(k+1)}(r + \Delta r) - V^{(k+1)}(r) \leq \max_p \left\{ \frac{a_1 \gamma}{b_1} \Delta r + \beta \frac{\gamma a_1}{(1-\beta \theta b_1)} \theta \Delta r \right\} \leq \frac{\gamma a_1}{(1-\beta \theta b_1)} \Delta r.$$ 

Hence the inequality $0 \leq V^{(k)}(r + \Delta r) - V^{(k)}(r) \leq \frac{\gamma a_1}{(1-\beta \theta b_1)} \Delta r$ holds for all $k \geq 0$ as well as for $V(r)$. This completes the proof.

**Proof of Proposition 4**: Similarly to the proof of Lemma 3, this proof uses the value iterations

$$V^{(0)}(r) = V(0)(r) \equiv 0,$$

$$V^{(k+1)}_m(r) = \Pi(p_m(r),r) + \beta V^{(k)}_m(\theta r + (1-\theta)p_m(r)),$$

$$V^{(k+1)}(r) = \Pi(p^*(r),r) + \beta V^{(k)}(\theta r + (1-\theta)p^*(r));$$

thus $V^{(\infty)}(r) = V(r)$ and $V^{(\infty)}_m(r) = V_m(r)$. Next we show (by induction) that $V^{(k)}(r) - V^{(k)}_m(r) \leq \frac{\beta(1-\theta)\lambda a_1^2}{(1-\beta)(1-\beta \theta b_1^2)}$. For $k = 0$, we have $V_m \equiv V$ and so the inequality holds. For $k \geq 0$,

$$V^{(k+1)}(r) - V^{(k+1)}_m(r) = \Pi(p^*(r),r) - \Pi(p_m(r),r) + \beta (V^{(k)}(\theta r + (1-\theta)p^*(r)) - V^{(k)}_m(\theta r + (1-\theta)p_m(r)))$$

$$\leq \beta (V^{(k)}(\theta r + (1-\theta)p^*(r)) - V^{(k)}_m(\theta r + (1-\theta)p_m(r)))$$

$$= \beta (V^{(k)}(\theta r + (1-\theta)p^*(r)) - V^{(k)}(\theta r + (1-\theta)p_m(r)))$$

$$+ \beta (V^{(k)}(\theta r + (1-\theta)p_m(r)) - V^{(k)}_m(\theta r + (1-\theta)p_m(r))));$$

(6)

here the inequality follows because $p_m(r)$ maximizes $\Pi(p,r)$. By Lemma 3, the first term of (6) is bounded by

$$(p^*(r) - p_m(r)) \frac{\beta(1-\theta)\lambda a_1}{(1-\beta \theta b_1^2)} \leq \frac{\beta(1-\theta)\lambda a_1^2}{(1-\beta \theta b_1^2)}.$$ 

According to the induction hypothesis, the second term of (6) is bounded by $\frac{\beta^2(1-\theta)\lambda a_1^2}{(1-\beta)(1-\beta \theta b_1^2)}$. Therefore,

$$V^{(k+1)}(r) - V^{(k+1)}_m(r) \leq \frac{\beta(1-\theta)\lambda a_1^2}{(1-\beta \theta b_1^2)} + \frac{\beta^2(1-\theta)\lambda a_1^2}{(1-\beta)(1-\beta \theta b_1^2)} = \frac{\beta(1-\theta)\lambda a_1^2}{(1-\beta \theta b_1^2)}.$$ 

Thus the inequality holds for all $k \geq 0$. Taking $k \to \infty$ now proves the result for $V_m(r) - V(r)$.

**Proof of Proposition 5**: We will prove the proposition by contradiction. Suppose a steady state $p_s$ exists. Then, by equation (2), $V(p_s) = \Pi(p_s,p_s) + \beta V(p_s)$ or (equivalently) $V(p_s) = \frac{\Pi(p_s,p_s)}{1-\beta}$. We will show that if $\gamma$ is sufficiently large then setting price $p_s$ is not optimal when $r = p_s$; that is, we can find another pricing policy that generates a higher discounted revenue than $\frac{\Pi(p_s,p_s)}{1-\beta}$.

For this purpose, we start by showing that $V(p_s)$ or $\Pi(p_s,p_s))$ has both an upper and a lower bound that are positive and independent of $\gamma$ and $\lambda$. The lower bound is given by $\Pi_1(a_2/2b_1)$, which is the optimal single-period revenue from segment 1. That revenue is clearly less than the total revenue of the optimal dynamic policy $V(p_s)$ and is also independent of $\gamma$ and $\lambda$. The upper bound of $\Pi(p_s,p_s)$ is given by $\Pi(p_s,p_s) = \max \{ p_s(a_1-b_1p_s), p_s(a_1+a_2-(b_1+b_2)p_s) \} \leq \frac{a_1^2}{4m} \frac{(a_1+a_2)^2}{(b_1+b_2)4}$, which is also independent of $\gamma$ and $\lambda$.

Define a continuous function $f(p) \equiv \Pi(p,p)$. Clearly, $f(p)$ is independent of both $\gamma$ and $\lambda$;
furthermore, \( f(0) = 0 \). Since \( f(p_s) = (1 - \beta)V(p_s) \) is bounded away from zero, it follows (by the continuity of \( f(p) \) at \( p = 0 \)) that \( p_s \) is outside a neighborhood \((0, \delta)\) of zero for some \( \delta > 0 \). Moreover, the choice of \( \delta \) is independent of \( \gamma \) and \( \lambda \).

Next consider the following pricing policy for \( r_0 = p_s \). Let \( p_k = \delta/2 \) for \( k \geq 0 \)—that is, charge a constant price \( \delta/2 \). Then the revenue generated at \( t = 0 \) is

\[
\Pi\left(\frac{\delta}{2}, p_s\right) \geq \frac{\delta}{2} \left(a_2 - b_2 \delta^2 + \gamma \left(p_s - \frac{\delta}{2}\right)\right) \geq \frac{\gamma \delta^2}{4} + \frac{\delta}{2} \left(a_2 - b_2 \frac{\delta}{2}\right).
\]

As \( \gamma \) increases, the RHS of this inequality can grow without bound. In particular, for a sufficiently large \( \gamma \) we have \( \Pi(\delta/2, p_s) > \Pi(p_s, p_s)/\left(1 - \beta\right) = V(p_s) \) because \( V(p_s) \) has an upper bound that is independent of \( \gamma \) and \( \lambda \). This outcome contradicts the optimality of \( V(p_s) \), thereby completing the proof.

**Proof of Theorem 4** Define a constant \( C_1 \triangleq \frac{1}{1 - \beta} \left(\frac{a_2^2}{s_1} + \frac{a_2^2}{s_2}\right) \). Our goal is to show that for \( \gamma > \max\left\{\frac{64C_1b_2^2}{a_1^2}, \frac{163 - K^2C_1b_2^2}{\delta^2a_1^2}, \frac{43 - K^2C_1b_2^2}{a_2^2}\right\} \), the optimal policy does not admit a steady state. We prove the claim by contradiction. Suppose the optimal price has a steady state, i.e., for all \( \epsilon > 0 \), we can find a \( T_1(\epsilon) > 0 \) so that \( \max_{t \geq T_1} p_t^* - \min_{t \geq T_1} p_t^* < \epsilon \). Consider a particular \( \epsilon = \min\left\{\frac{a_1}{b_1}, \frac{a_1}{s_1}\right\} \) and such \( T_1 \). Without loss of generality, for \( t \geq T_1 \), we can find \( \bar{p} \) such that \( |p_t^* - \bar{p}| < \epsilon/2 \). Because the reference formation process is asymptotically oblivious, for any small number \( \epsilon_1 > 0 \) we can find \( T_2 > T_1 \), such that for \( t \geq T_2 \), we have \( \sum_{i=0}^{T_1} w_{t,i} \epsilon_1 < \epsilon_1 \). In particular, we can choose \( \epsilon_1 \) sufficiently small so that

\[
r_t^* = w_{t-1}r_{t-1}^* + \sum_{i=0}^{t} w_{t,i}p_t^* \geq \sum_{i=T_1+1}^{t} w_{t,i}p_t^* \geq (\bar{p} - \epsilon/2)(1 - \epsilon_1) \geq \bar{p} - \epsilon \quad \text{and}
\]

\[
r_t^* \leq \epsilon_1 a_1/b_1 + \sum_{i=T_1+1}^{t} w_{t,i}p_t^* \leq \bar{p} + \epsilon/2 + \epsilon_1 a_1/b_1 \leq \bar{p} + \epsilon,
\]

where \( r_t^* \) is the reference price associated with the policy \( p_t^* \). Therefore, for \( t \geq T_2 \), we have \( |p_t^* - \bar{p}| < \epsilon/2 \) and \( |r_t^* - \bar{p}| < \epsilon \).

Consider the revenues generated by the optimal policy from \( T_2 \) onward, discounted to period \( T_2 \): \( \sum_{i=0}^{\infty} \beta^i \Pi(p_{T_2+i}^*, r_{T_2+i}^*) \). By the above results, we have

\[
\sum_{i=0}^{\infty} \beta^i \Pi(p_{T_2+i}^*, r_{T_2+i}^*) \leq \sum_{i=0}^{\infty} \beta^i \left(p_{T_2+i}^* D_1(p_{T_2+i}^*) + p_{T_2+i}^* D_2(p_{T_2+i}^*, r_{T_2+i}^*)\right)
\]

\[
\leq \sum_{i=0}^{\infty} \beta^i \left(\frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2} + \frac{a_1}{b_1} \gamma (r_{T_2+i}^* - p_{T_2+i}^*)\right)
\]

\[
\leq \frac{1}{1 - \beta} \left(\frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2} + \frac{3a_1 \gamma \epsilon}{2b_1}\right)
\]

\[
\leq \frac{1}{1 - \beta} \left(\frac{a_1^2}{2b_1} + \frac{a_2^2}{4b_2}\right) = C_1
\]

where the last inequality holds because \( \gamma \epsilon \geq a_1/6 \).

We next show that \( \bar{p} + \epsilon < a_1/2b_1 \). Combined with the previous result that \( |p_t^* - \bar{p}| < \epsilon/2 \), this will imply that the steady state of the optimal policy is below \( a_1/2b_1 \). If this is not true, then we
have $r_{T_2}^* \geq \tilde{p} - \epsilon \geq a_1/2b_1 - 2\epsilon \geq a_1/4b_1$ (because $\epsilon \leq a_1/8b_1$). Consider the following policy $p'_t$: from $t = 0$ to $T_2 - 1$, $p'_t = p_t^*$; for $t = T_2$, $p'_t = \min\{a_1/8b_1, a_2/2b_2\}$; for $t \geq T_2 + 1$, $p'_t = 0$. Clearly, the two policies $p_t^*$ and $p'_t$ generate the same revenue before period $T_2$. Now consider the revenue generated in $T_2$ for the policy $p'_t$. Because $p_{T_2}' \leq r_{T_2}' = r_{T_2}'$, we have

$$\Pi(p_{T_2}', r_{T_2}') \geq p_{T_2}' \left( a_2 - b_2p_{T_2}' + \gamma \left( \frac{a_1}{4b_1} - p_{T_2}' \right) \right)^+. \quad (7)$$

If $a_1/8b_1 \leq a_2/2b_2$, then $p_{T_2}' = a_1/8b_1$ and

$$\text{RHS of (7)} \geq \frac{a_1}{8b_1} \left( \gamma \frac{a_1}{8b_1} \right) \geq \gamma \frac{a_1^2}{64b_1^2} > C_1.$$

If $a_1/8b_1 > a_2/2b_2$, then $p_{T_2}' = a_2/2b_2$ and

$$\text{RHS of (7)} \geq \frac{a_2}{2b_2} \left( \gamma \frac{a_2}{2b_2} \right) \geq \gamma \frac{a_2^2}{4b_2^2} > C_1.$$

Either way, the revenue generated in a single period $T_2$ by the new policy $p'_t$ is higher than $C_1$, which is the total discounted revenue generated by the optimal policy from $T_2$ onward. This contradicts the fact that $p_t^*$ is optimal. Thus, we have proved that $p_t^*$ is optimal.

We next show that $p_t^*$ cannot be optimal. Consider the following policy $p'_t$: from $t = 0$ to $T_2 - 1$, $p'_t = p_t^*$; for $t = T_2$ to $T_2 + K - 1$, $p'_t = a_1/b_1$; for $t = T_2 + K$, $p'_t = \min\{a_2/2b_2, \delta a_1/4b_1\}$; for $t \geq T_2 + K + 1$, $p'_t = 0$. Now we compute the revenue generated in period $T_2 + K$. Because $p_t^* \leq \tilde{p} + \epsilon < a_1/2b_1$ for $t = T_2, \ldots, T_2 + K - 1$, we have

$$r_{T_2 + K}' = w_{T_2 + K, -1}r_0 + \sum_{i=0}^{T_2-1} w_{T_2 + K, i}p_t^* + \sum_{i=T_2}^{T_2 + K - 1} w_{T_2 + K, i} \left( p_t^* + \frac{a_1}{2b_1} \right)$$

$$\geq w_{T_2 + K, -1}r_0 + \sum_{i=0}^{T_2-1} w_{T_2 + K, i}p_t^* + \sum_{i=T_2}^{T_2 + K - 1} w_{T_2 + K, i} \left( p_t^* + \frac{a_1}{2b_1} \right)$$

$$\geq r_{T_2 + K}' + \frac{a_1}{2b_1} \sum_{i=T_2}^{T_2 + K - 1} w_{T_2 + K, i} \geq r_{T_2 + K}^* + \frac{a_1\delta}{2b_1}.$$

The last inequality follows from the fact that the reference formation is $(K, \delta)$-retentive. Then the revenue in period $T_2 + K$ is

$$\Pi(p_{T_2 + K}', r_{T_2 + K}') \geq p_{T_2 + K}' \left( a_2 - b_2p_{T_2 + K}' + \gamma \left( \frac{a_1\delta}{2b_1} - p_{T_2 + K}' \right) \right). \quad (8)$$

If $a_2/2b_2 \leq \delta a_1/4b_1$, then $p_{T_2 + K}' = a_2/2b_2$ and

$$\text{RHS of (8)} \geq \gamma \frac{a_2^2}{4b_2^2} > \beta^{-K}C_1.$$
If \( \frac{a_2}{2b_2} > \frac{\delta a_1}{4b_1} \), then \( p_{T_2 + K}' = a_1 \delta / 4b_1 \) and
\[
\text{RHS of (8)} \geq \gamma \frac{\delta^2 a_1^2}{16b_1^2} > \beta^{-K} C_1.
\]

In either way, the revenue generated by \( p_t' \) in period \( T_2 + K \), discounted to \( T_2 \), is higher than \( C_1 \), the total discounted revenue generated by the optimal policy from \( T_2 \) onward. Because \( p_t' \) and \( p_t^* \) generate the same revenue before \( T_2 \), it contradicts the fact that \( p_t^* \) is the optimal policy. Therefore, we have proved that \( p_t^* \) does not admit a steady state. ■

## B. Additional Results and Their Proofs

### B.1. Periodic Markdown

In this appendix, we provide an analytical characterization of the periodic markdown policy which we discussed in Section 2.1. Assume w.l.o.g. that the firm offers a discount at the end of the cycle. Let \( p_m(r_0) = p_m(r_1) = \cdots = p_m(r_{n-2}) = \frac{a_1}{2b_1} \) (the firm optimizes its revenue from segment 1) and \( p_m(r_{n-1}) = \frac{a_1 + a_2 + \gamma r_{n-1}}{2(2b_1 + b_2 + \gamma)} \) (the firm optimizes its revenue from both segments). By the evolution of the reference price, we have
\[
\begin{align*}
r_1 &= \theta r_0 + (1 - \theta) \frac{a_1}{2b_1}, \\
\vdots \\
r_{n-1} &= \theta r_{n-1} + (1 - \theta) \frac{a_1}{2b_1} = \theta^{n-1} r_0 + (1 - \theta^{n-1}) \frac{a_1}{2b_1}, \\

r_n &= \theta r_{n-1} + (1 - \theta) \frac{a_1 + a_2 + \gamma r_{n-1}}{2(2b_1 + b_2 + \gamma)}.
\end{align*}
\]

Because \( r_n = r_0 \), we can use these equations to solve \( r_0, r_1, \ldots, r_{n-1} \). In order for a periodic markdown pricing policy to be an optimal myopic policy, we require \( r_i \leq \bar{r}^\gamma \) for \( i = 0, \ldots, n-2 \) so that \( p_m(r_i) = \frac{a_1}{2b_1} \), and \( r_{n-1} > \bar{r}^\gamma \) so that \( p_m(r_{n-1}) = \frac{a_1 + a_2 + \gamma r_{n-1}}{2(2b_1 + b_2 + \gamma)} \) (see case (iii) of Proposition 1). These conditions lead to our next proposition.

**Proposition B.1.** Suppose \( \bar{r}^\gamma \in (r^\gamma, a_1/2b_1) \). Then the myopic pricing policy is a periodic markdown policy with cycle length \( n \) if and only if
\[
\theta^{n-2} r_0 + (1 - \theta^{n-2}) \frac{a_1}{2b_1} < \bar{r}^\gamma < \theta^{n-1} r_0 + (1 - \theta^{n-1}) \frac{a_1}{2b_1};
\]
here \( r_0 = \frac{b(1-\theta^{n-1})(a_1/2b_1)+(1-\theta)l}{1-\theta^{n-1}b} \) is the initial reference price, \( b = \theta + (1 - \theta) \frac{\gamma}{2(b_1 + b_2 + \gamma)} \), and \( l = \frac{a_1 + a_2}{2(b_1 + b_2 + \gamma)} \).

The periodic markdown policy depends only on perceived gains and not on perceived losses—for the same reason discussed after Proposition 2. Moreover, the parameter values that give rise to cyclic pricing (case (iv) of Proposition 2) may not satisfy the condition in Proposition B.1 for some \( n \). Such circumstances may cause the emergence of cyclic behavior more complex than periodic markdowns.

As \( \gamma \) increases, \( \bar{r}^\gamma \) decreases and \( r_{n-1} > \bar{r}^\gamma \) tends to be satisfied for smaller \( n \). This implies that a stronger reference effect \( \gamma \) may lead to more frequent discounts. The reason is that the firm finds it more profitable to price-skim segment 2 for a larger \( \gamma \), which it should do more frequently.
even if the reference price is not very high. This approach does not run afoot of the bifurcation phenomenon (see text following Proposition 3) because we address only periodic markdown policies and ignore cyclic pricing policies of greater complexity.

To illustrate, let \( a_1 = 1, b_1 = 0.1, a_2 = 1.5, b_2 = 2, \gamma = 0.7, \lambda = 0.75, \theta = 0.5, \) and \( r_0 = 2.71. \) The cycle length is 3.

**Figure 1:** Reference price path (blue dashed lines) and optimal myopic pricing path (red dashed lines) for \( a_1 = 1, b_1 = 0.1, a_2 = 1.5, b_2 = 2, \gamma = 0.7, \lambda = 0.75, \theta = 0.5, \) and \( r_0 = 2.71. \) The cycle length is 3.

### B.2. Peak-end Rule

The ‘peak-end rule’ is a recursive reference point formation mechanism which assumes consumers anchor on the most recent and the minimum prices paid for a product (Nasiry and Popescu, 2011), or

\[
r_t = \theta m_{t-1} + (1 - \theta)p_{t-1}
\]

where \( m_{t-1} = \min(m_{t-2}, p_{t-1}) \).

Because the peak-end rule assigns a positive weight on the minimum price, it cannot be represented by deterministic weights. Nevertheless, we can still show that for sufficiently large \( \gamma \), the firm’s optimal policy does not admit a steady state.

**Theorem B.1.** Assume the reference point follows the peak-end rule. Then the optimal pricing policy \( p_t^* \) for \( t = 1, 2, \ldots \) does not admit a steady state for a sufficiently large \( \gamma \). That is, there exists \( \epsilon > 0 \) such that for all \( T \geq 1 \),

\[
\max_{t \geq T} p_t^* - \min_{t \geq T} p_t^* > \epsilon.
\]
B.3. Proofs

Proof of Proposition B.1. Periodic markdown is an instance of case 3 in Proposition 1 or of case (iv) in Proposition 2. So for \( p_m(r) \) as derived in Proposition 1, we may write

\[
  r_1 = \theta r_0 + (1 - \theta) \frac{a_1}{2b_1},
\]

\[
  r_{n-1} = \theta r_{n-1} + (1 - \theta) \frac{a_1}{2b_1} = \theta^{n-1} r_0 + (1 - \theta^{n-1}) \frac{a_1}{2b_1},
\]

\[
  r_n = \theta r_{n-1} + (1 - \theta) \frac{a_1 + a_2 + \gamma r_{n-1}}{2(b_1 + b_2 + \gamma)} = \left( \theta + (1 - \theta) \frac{\gamma}{2(b_1 + b_2 + \gamma)} \right) \left( \theta^{n-1} r_0 + (1 - \theta^{n-1}) \frac{a_1}{2b_1} \right) + \frac{(1 - \theta)(a_1 + a_2)}{2(b_1 + b_2 + \gamma)} = r_0.
\]

From the last of these equalities it follows that \( r_0 = \frac{b(1 - \theta^{n-1})(a_1/2b_1) + (1 - \theta)l}{1 - \theta^{n-1}b} \), where \( b = \theta + (1 - \theta) \frac{\gamma}{2(b_1 + b_2 + \gamma)} \) and \( l = \frac{a_1 + a_2}{2(b_1 + b_2 + \gamma)} \).

For periodic markdown to be a myopically optimal pricing policy, it must match \( p_m(r) \) in case 3 of Proposition 1. The implication is that \( r_{n-2} < \bar{r} \) and also \( r_{n-1} > \bar{r} \). Therefore,

\[
  r_{n-2} < \bar{r} < r_{n-1} \iff \theta^{n-2} r_0 + (1 - \theta^{n-2}) \frac{a_1}{2b_1} < \bar{r} < \theta^{n-1} r_0 + (1 - \theta^{n-1}) \frac{a_1}{2b_1},
\]

where we have used the expression \( r_i = \theta^i r_0 + (1 - \theta^i) \frac{a_1}{2b_1} \) for \( i \leq n-2 \). This completes the proof. \( \blacksquare \)

Proof of Theorem B.1. Because the peak-end rule cannot be represented by (3), we cannot directly use the same method as in the proof of Theorem 1. The key difference is that, for a pricing policy that converges to a steady state, the reference price may converge to a different steady state. This does not turn out to be a problem as the peak-end rule implies that the steady state of the reference price is always lower than the steady state price. Therefore, we can use a similar perturbation approach used in Theorem 1.

Define a constant \( C_1 \triangleq \frac{1}{1-\beta} \left( \frac{a_1^2}{2b_1} + \frac{a_2^2}{4b_2} \right) \). We will show that for \( \gamma > \max \left\{ \frac{64C_1 b_1^2}{\beta(1-\theta)^2 a_1}, \frac{4C_1 b_2^2}{\gamma \beta a_2^2} \right\} \), the optimal policy does not admit a steady state.

We prove the claim by contradiction. Suppose the optimal price has a steady state, i.e., for all \( \epsilon > 0 \), we can find a \( T_1(\epsilon) > 0 \) so that \( \max_{t \geq T_1} p_t^* - \min_{t \geq T_1} p_t^* < \epsilon \). Consider a particular \( \epsilon = \min \left\{ \frac{a_1}{8(1-\theta)b_1}, \frac{a_1}{8(1-\theta)b_1} \right\} \) and such \( T_1 \). Without loss of generality, for \( t \geq T_1 \), we can find \( \bar{p} \) such that \( |p_t^* - \bar{p}| < \epsilon/2 \). Denote \( \bar{p} = \min_{1 \leq t \leq T_1} \{ p_t^* \} \). Clearly, \( \bar{p} \leq p_{T_1}^* \leq \bar{p} + \epsilon/2 \). Therefore, for \( t \geq T_1 + 2 \)

\[
  r_t^* = (1 - \theta) p_{t-1}^* + \theta \min \left\{ \bar{p}, p_{T_1+1}^*, \ldots, p_{t-1}^* \right\} \leq (1 - \theta) \bar{p} + \theta \bar{p} + \frac{\epsilon}{2}
\]

and

\[
  r_{t}^* = (1 - \theta) p_{t-1}^* + \theta \min \left\{ \bar{p}, \bar{p} - \frac{\epsilon}{2} \right\} \geq (1 - \theta) \bar{p} + \theta \bar{p} - \epsilon,
\]

where \( r_t^* \) is the reference prices associated with the policy \( p_t^* \). Therefore, for \( t \geq T_1 + 2 \), we have \( |p_t^* - \bar{p}| < \epsilon/2 \) and \( |r_t^* - (1 - \theta) \bar{p} - \theta \bar{p}| < \epsilon \). As a result, \( r_t^* \leq (1 - \theta) \bar{p} + \theta \bar{p} + \epsilon \leq \bar{p} + 3\epsilon/2 \leq p_t^* + 2\epsilon \); and \( r_t^* \geq (1 - \theta) \bar{p} + \theta \bar{p} - \epsilon \geq (1 - \theta)(p_t^* - \epsilon/2) - \epsilon \geq (1 - \theta)p_t^* - 2\epsilon \).

Consider the revenues generated by the optimal policy from \( T_1 + 2 \) onward, discounted to period
\[ T_1 + 2: \sum_{i=0}^{\infty} \beta^i \Pi(p_{T_1+2+i}^*, r_{T_1+2+i}^*) \text{.} \] By the above results, we have
\[
\sum_{i=0}^{\infty} \beta^i \Pi(p_{T_1+2+i}^*, r_{T_1+2+i}^*) \leq \sum_{i=0}^{\infty} \beta^i \left( p_{T_1+2+i}^* D_1(p_{T_1+2+i}^*) + p_{T_1+2+i}^* D_2(p_{T_1+2+i}^*, r_{T_1+2+i}^*) \right)
\leq \sum_{i=0}^{\infty} \beta^i \left( \frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2} + \frac{a_1}{b_1} \gamma (r_{T_1+2+i}^* - p_{T_1+2+i}^*) \right)
\leq \frac{1}{1 - \beta} \left( \frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2} + \frac{2a_1 \gamma \epsilon}{b_1} \right)
\leq \frac{1}{1 - \beta} \left( \frac{a_1^2}{2b_1} + \frac{a_2^2}{4b_2} \right) = C_1
\]
where the last inequality is by the fact that \( \gamma \epsilon \leq a_1 / 8 \).

We next show that \( r_{T_1+2}^* \leq (1 - \theta) a_1 / 2b_1 \). Otherwise, consider the following policy \( p_t' \) from \( t = 0 \) to \( T_1 + 1 \), \( p_t' = p_{t1}^* \); for \( t = T_1 + 2 \), \( p_t' = \min \{ (1 - \theta) a_1 / 4b_1, a_2 / 2b_2 \} \); for \( t \geq T_1 + 3 \), \( p_t' = 0 \). Clearly, the two policies \( p_t^* \) and \( p_t' \) generate the same revenue before period \( T_1 + 2 \). Now consider the revenue generated in \( T_1 + 2 \) for the policy \( p_t' \). Because \( p_t' \leq r_{T_1+2}^* = r_{T_1+2}' \), we have
\[
\Pi(p_{T_1+2}', r_{T_1+2}') \geq p_{T_1+2}' \left( a_2 - b_2 p_{T_1+2}' + \gamma \left( \frac{(1 - \theta) a_1}{2b_1} - p_{T_1+2}' \right) \right) \quad (9)
\]
If \( (1 - \theta) a_1 / 4b_1 \leq a_2 / 2b_2 \), then \( p_{T_1+2}' = (1 - \theta) a_1 / 4b_1 \) and
\[
\text{RHS of (9)} \geq \frac{(1 - \theta) a_1}{4b_1} \left( \gamma \frac{(1 - \theta) a_1}{4b_1} \right) \geq \gamma \frac{(1 - \theta)^2 a_1^2}{16b_1^2} > C_1.
\]
If \( (1 - \theta) a_1 / 4b_1 > a_2 / 2b_2 \), then \( p_{T_1+2}' = a_2 / 2b_2 \) and
\[
\text{RHS of (9)} \geq \frac{a_2}{2b_2} \left( \gamma \frac{a_2}{2b_2} \right) \geq \gamma \frac{a_2^2}{4b_2^2} > C_1.
\]
In either way, the revenue generated in a single period \( T_1 + 2 \) by the new policy \( p_t' \) is higher than \( C_1 \), which is the total discounted revenue generated by the optimal policy from \( T_1 + 2 \) onward. This contradicts the fact that \( p_t^* \) is optimal. Thus, we have proved that \( r_{T_1+2}^* \leq (1 - \theta) a_1 / 2b_1 \). Therefore, \( p_t^* \leq r_{T_1+2}/(1 - \theta) + 2\epsilon/(1 - \theta) \leq a_1 / 2b_1 + 2\epsilon/(1 - \theta) \leq 3a_1 / 4b_1 \).

We next show that \( p_t^* \) cannot be optimal. Consider the following policy \( p_t' \) from \( t = 0 \) to \( T_1 + 1 \), \( p_t' = p_{t1}^* \); for \( t = T_1 + 2 \), \( p_t' = a_1 / b_1 \); for \( t = T_1 + 3 \), \( p_t' = \min \{ (1 - \theta) a_1 / 8b_1, a_2 / 2b_2 \} \); for \( t \geq T_1 + 4 \), \( p_t' = 0 \). Now we compute the revenue generated in period \( T_1 + 3 \). We have
\[
r_{T_1+3}' = (1 - \theta) p_{T_1+2}' + \min_{0 \leq t \leq T_1+2} \{ p_t' \}
\geq (1 - \theta) p_{T_1+2}^* + (1 - \theta) \left( \frac{a_1}{b_1} - p_{T_1+2}^* \right) + \min_{0 \leq t \leq T_1+2} \{ p_t^* \}
\geq r_{T_1+3}^* + \frac{(1 - \theta) a_1}{4b_1}.
\]
Then the revenue in period $T_1 + 3$ is

$$\Pi(p'_{T_1+3}, r'_{T_1+3}) \geq p'_{T_1+3} \left( a_2 - b_2 p'_{T_1+3} + \gamma \left( \frac{(1 - \theta)a_1}{4b_1} - p'_{T_1+3} \right) \right).$$

(10)

If $a_2/2b_2 \leq (1 - \theta)a_1/8b_1$, then $p'_{T_1+3} = a_2/2b_2$ and

$$\text{RHS of (10)} \geq \gamma \frac{a_2^2}{4b_2} > \beta^{-1} C_1.$$

If $a_2/2b_2 > (1 - \theta)a_1/8b_1$, then $p'_{T_1+3} = (1 - \theta)a_1/8b_1$ and

$$\text{RHS of (10)} \geq \gamma \frac{(1 - \theta)^2 a_1^2}{64b_1^2} > \beta^{-1} C_1.$$

Either way, the revenue generated by $p'_t$ in period $T_1 + 3$, discounted to $T_1 + 2$, is higher than $C_1$, the total discounted revenue generated by the optimal policy from $T_1 + 2$ onward. Because $p'_t$ and $p^*_t$ generate the same revenue before $T_1 + 2$, it contradicts the fact that $p^*_t$ is the optimal policy. Therefore, we have proved that $p^*_t$ does not admit a steady state.

**References**
